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A Tale of Two Arrows

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Two time arrows for scattering processes have been proposed in rigged Hilbert space quantum mechanics. One, due to Arno Bohm, involves preparations and registrations in laboratory operations and results in two semigroups oriented in the forward direction of time. The other, employed by the Brussels–Austin group, is more general, involving excitations and de-excitations of systems, and apparently results in two semigroups oriented in opposite directions of time. The relationship between these two arrows is discussed.

KEY WORDS: time arrows; Hilbert space.

1. TWO TIME ARROWS

In the standard formulation of nonrelativistic quantum mechanics, the time evolution of systems is governed by a one-parameter group of unitary operators

$$U(t) = e^{-iHt} \tag{1}$$

on a Hilbert space (HS) (von Neumann, 1955/1932), where H represents the Hamiltonian and Planck's constant has been set to one. Any evolution governed by (1) is time-reversal invariant[2] and irreversibility[3] usually enters in because of an extrinsic act of measurement or other interaction with an environment (Zeh, 1999). This approach, however, does not allow for intrinsic forms of irreversibility, where irreversible behavior originates in the dynamics of a physical system without explicit reference to an environment (Atmanspacher and Bishop, 2002). Such irreversible behavior cannot be appropriately modeled nor can appropriate initial conditions for such irreversible processes be formulated rigorously in HS. For these, among other reasons (Bishop, in press; Bohn *et al.*, 1997), theories of rigged Hilbert space (RHS) quantum mechanics—a generalization of the HS

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[2] Time-reversal invariance means that if $\phi(t)$ is a solution of the quantum mechanical equations of motion, then so is $\phi(-t)$.

[3] A process is *reversible* if the temporal succession of its states $\phi_1, \phi_2, \ldots, \phi_n$ can occur in the opposite order and *irreversible* otherwise.

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version—were developed (Antoniou and Prigogine, 1993; Bohn *et al.*, 1997). An RHS, or Gel'fand triplet (Gel'fand and Shilov, 1967; Gel'fand and Vilenkin, 1964), is the tirple of spaces

$$\phi \subset \mathcal{H} \subset \Phi^{\times}, \tag{2}$$

where $\mathcal{H}$ is a HS with the standard norm topology, $\tau_{\mathcal{H}}$, Φ is a vector space with a topology, τ_{Φ}, stronger than $\tau_{\mathcal{H}}$, and $\Phi^{\times}$ is the dual space of continuous linear functionals on Φ.

In the context of scattering theory, two intrinsic arrows of time have been proposed within RHS quantum mechanics. One, due to Bohm (Bohm *et al.*, 1997; Bohm and Gadella, 1989), involves preparations and registrations in laboratory operations, resulting in semigroups oriented in the forward direction of time. The key intuition behind this arrow is that no observable properties of a state can be measured unless the state has first been prepared. Prepared in-states are taken to be elements $\phi \in \Phi_{-}$ and observables (so-called out-states of postinteraction particles) are taken to be elements $\psi \in \Phi_{+}$ (Decaying states, Gamow vectors, are elements of $\Phi^{\times}_{\pm}$.) This leads to a distinction between prepared states and observables, each described by a separate RHS:

$$\Phi_{-} \subset \mathcal{H} \subset \Phi^{\times}_{-}, \tag{3a}$$

$$\Phi_{+} \subset \mathcal{H} \subset \Phi^{\times}_{+}, \tag{3b}$$

where Φ_{-} is the Hardy space of the lower complex energy half-plane intersected with the Schwartz class functions and Φ_{+} is the Hardy space of the upper complex energy half-plane intersected with the Schwartz class functions. Some elements of the generalized eigenstates in $\Phi^{\times}_{-}$ and $\Phi^{\times}_{+}$ correspond to exponentially growing and decaying states respectively (Bohn *et al.*, 1997; Bohn and Wickramasekara, 1997). The semigroups governing these states are[4]

$$\langle\phi|U^{\times}|Z_R^*\rangle = e^{-iE_Rt}e^{\frac{\Gamma}{2}t}\langle\phi|Z_R^*\rangle \qquad t \leq 0, \quad t: -\infty \rightarrow 0, \tag{4a}$$

$$\langle\phi|U^{\times}|Z_R\rangle = e^{-iE_Rt}e^{-\frac{\Gamma}{2}t}\langle\psi|Z_R\rangle \qquad t \geq 0, \quad t: 0 \rightarrow \infty, \tag{4b}$$

where E_R represents the total resonance energy, Γ represents the resonance width, Z_R represents the pole at $E_R - i\frac{\Gamma}{2}$, Z_R^* represents the pole at $E_R + i\frac{\Gamma}{2}$, $|Z_R^*\rangle \in \Phi^*_{-}$ represents a growing Gamow vector, and $|Z_R\rangle \in \Phi^{\times}_{+}$ represents a decaying Gamow vector. The $t < 0$ semigroup is identified as future-directed along with $|Z_R^*$ as a forming/growing state. The $t > 0$ semigroup is identified as future-directed along with $|Z_R\rangle$ as a decaying state.[5]

[4] When the group operator U is extended to $\Phi^{\times}$, continuity requirements force the extended operators to be semigroups defined only on the temporal half-domains (Bohn *et al.*, 1997; Bohn and Wickramasekara, 1997).

[5] Note that the eigenvectors plus the semigroup property are insufficient to determine the temporal direction of evolution. These identifications involve further physical considerations.

The other time arrow, originally proposed by George (1971) and employed by the Brussels–Austin group, is more general involving excitations and de-excitations of systems, resulting in semigroups apparently oriented in opposite directions of time. In their discussion of scattering and resonance phenomena, Antoniou and Prigogine (1993) also apply the RHS framework, using the Hardy class functions as a natural function space for their analysis. Antoniou and Prigogine adopt the following time arrow: excitations are interpreted as events taking place before $t = 0$ whereas de-excitations are interpreted as events taking place after $t = 0$. This time arrow leads to a natural splitting of the RHS: excitations (e.g., formation of unstable states) are considered as past-oriented and associated with $\phi_+ \in \Phi_+^\times$ in the upper half-plane, while de-excitations (e.g. decay of unstable states) are considered as future-oriented and associated with $\phi_- \in \Phi_-^\times$ in the lower half-plane.[6] The semigroups governing decaying states as identified by the Brussels–Austin group are

$$\langle \phi_+ | U^\times | Z_R^* \rangle = e^{iE_R t} e^{\frac{\Gamma}{2} t} \langle \phi_+ | Z_R^* \rangle \qquad t < 0, \quad t : -\infty \leftarrow 0 \tag{5a}$$

$$\langle \phi_- | U^\times | Z_R \rangle = e^{-iE_R t} e^{-\frac{\Gamma}{2} t} \langle \phi_- | Z_R \rangle \qquad t > 0, \quad t : 0 \leftarrow \infty. \tag{5b}$$

The Brussels–Austin Group identifies the $t < 0$ semigroup as evolving states into the past along with $|Z_R^*\rangle$ as decaying states, and the $t > 0$ semigroup as evolving states into the future along with $|Z_R\rangle$ as decaying states.

2. TIME-REVERSED STATES AND OBSERVABLES

Following Wigner (1964), Bohm and coworkers have applied the time-reversal operator, $R(t)$, from the extended spacetime group to the states (4) for elastic scattering (Bohm, 1995; Bohm and Wickramasekara, 1997). The extended group contains four representations, the first leaving the underlying vector space unchanged (Wigner, 1964). This is the typical case discussed in quantum mechanics. The other three representations, however, exhibit a doubling of the vector spaces. To track this space doubling, let the index $r = 0, 1$ label the original vector space and its double, respectively. Then, applying the transformation properties of $R : \Phi_\pm^{r=0,\times} \to \Phi_\mp^{r=1,\times}$ (Bohm, 1995; Bohm and Wickramasekara, 1997), and identifying $r = 0$ with the normal scattering experiment and $r = 1$ with the time-reversed situation, $U^\times(t)\langle \phi, r = 0 | Z_R^*, r = 0 \rangle \in \Phi_-^{r=0,\times}$, a growing Gamow vector representing a preparable state for $t \leq 0$, is transformed under R

[6] Note that the roles of the upper and lower Hardy class function spaces are reversed with respect to Bohm's approach. This has only mathematical import. The differences in phase factors between (4) and (5) are due to the fact that in (4), states evolve in the Schrödinger picture while observables evolve in the Heisenberg picture, whereas in (5), only states evolve.

Table I. Properties of the Bohm/Gadella Gamow Vectors Under $R(t)$

Growing vectors	$\langle\phi, r=0\|Z_R^*, r=0\rangle$ $t \leq 0, t : -\infty \to 0$	$\langle\psi, r=1\|Z_R, r=1\rangle$ $t \geq 0, t : 0 \leftarrow \infty$
Decaying vectors	$\langle\psi, r=0\|Z_R, r=0\rangle$ $t \geq 0, t : 0 \to \infty$	$\langle\phi, r=1\|Z_R^*, r=1\rangle$ $t \leq 0, t : -\infty \leftarrow 0$

into $U^\times(-t)\langle\psi, r=1|Z_R, r=1\rangle \in \Phi_+^{r=1,\times}$, where

$$e^{iE_{Rt}}e^{-\frac{\Gamma}{2}t}\langle\psi, r=1|Z_R, r=1\rangle \tag{6}$$

is restricted to the time domain $t \geq 0$ by continuity requirements. In the case of $|Z_R^*, r=0\rangle$, time runs from $-\infty$ to 0; in contrast, for $|Z_R, r=1\rangle$, time runs from ∞ to 0, meaning that it represents a Gamow vector that grows as t decreases. Similarly, $U^\times(t)\langle\psi, r=0|Z_R, r=0\rangle \in \Phi_+^{r=0,\times}$, a decaying Gamow vector representing observables for $t \geq 0$, is transformed under R into $U^\times(-t)\langle\phi, r=1|Z_R^*, r=1\rangle \in \Phi_-^{r=1,\times}$, where

$$e^{iE_Rt}e^{\frac{\Gamma}{2}t}\langle\phi, r=1|Z_R^*, r=1\rangle \tag{7}$$

is restricted to the time domain $t \leq 0$ by continuity requirements. In the case of $|Z_R, r=0\rangle$, time runs from 0 to ∞; in contrast, for $|Z_R^*, r=1\rangle$, time runs from 0 to $-\infty$, meaning that it represents a Gamow vector that decays as $-t$ increases. These results are summarized in Table I.

Using the transformation rules as appropriate, the temporal evolution of the time-reversed vectors in the Brussels–Austin approach can be determined. However, notice that the eigenvectors in (5) are identified with decaying states. It can be easily seen that (5b) is the time-reversal of (5a) under R, but the label r associated with vector space doubling remains to be identified. If we assume that the preparation/registration arrow is a special case of the excitation/de-excitation arrow (Bishop, in press), then (5a) can be identified with the $r=1$ and (5b) with the $r=0$ regimes respectively.

What remains is to examine the eigenvectors representing growing states in the Brussels–Austin approach. To each de-excitation in (5) there is a corresponding excitation represented by an eigenvector in the opposite temporal half-plane. For the $r=0$ regime, a growing eigenvector of the form

$$e^{iE_Rt}e^{\frac{\Gamma}{2}t}\langle\phi_+, r=0|Z_R^*, r=0\rangle \tag{8}$$

corresponds to eigenstate (5b), where (8) is restricted to the time domain $t < 0$ by continuity requirements. This state is represented by a Gamow vector that grows as $-t$ decreases. Similarly, for the $r=1$ regime, a growing eigenstate of the form

$$e^{-iE_Rt}e^{-\frac{\Gamma}{2}t}\langle\phi_-, r=1|Z_R, r=1\rangle \tag{9}$$

Table II. Properties of the Brussels–Austin Gamow Vectors Under $R(t)$

Growing vectors	$\langle\phi_+, r=0 \vert Z_R^*, r=0\rangle$ $t<0, t: -\infty \to 0$	$\langle\phi_-, r=1 \vert Z_R, r=1\rangle$ $t>0, t: 0 \leftarrow \infty$
Decaying vectors	$\langle\phi_-, r=0 \vert Z_R, r=0\rangle$ $t>0, t: 0 \to \infty$	$\langle\phi_+, r=1 \vert Z_R^*, r=1\rangle$ $t<0, t: -\infty \leftarrow 0$

corresponds to eigenvector (5a), where (9) is restricted to the time domain $t > 0$ by continuity requirements. This state is represented by a Gamow vector that grows as t decreases. These results are summarized in Table II.

The Bohm and Brussels–Austin groups, then, were working with the same eigenvectors and semigroups in their analysis of scattering. Equations (5a) and (5b) are time-reversed images of each other and, when paired with their corresponding growing vectors, are easily related to those of Bohm and coworkers, which is not immediately apparent when comparing (4) and (5).

3. THE POSSIBILITY OF TIME-REVERSED STATES

Lee (1981) has discussed the formation of time-reversed quantum states for a $\bar{\mu}$-meson at rest with its spin $\mathbf{s}_\mu$ in the updirection, decaying as

$$\bar{\mu} \to e^- + \bar{\nu}_e + \nu_\mu, \tag{10}$$

where the electron, electron antineutrino, and μ neutrino are emitted with helicities $-1/2$, $1/2$, and $-1/2$ respectively. Producing a final state with $\bar{\mu}$ at rest and a final spin $\mathbf{s}'_\mu = -\mathbf{s}_\mu$ is not generally possible, requiring the momentum and spin of all three leptons be simultaneously reversed in all possible directions while maintaining the appropriate phase relations among their wave amplitudes. The latter assumes the creation of three perfectly coherent incoming spherical waves in the midst of the many degrees of freedom involved. Producing such a state in laboratory situations (preparation/registration arrow) is clearly impossible due to the precision required to produce such coherent incoming spherical waves, as well as the control over the environment it entails. For the more general case (excitation/de-excitation arrow), it is not clear that time-reversed growing states associated with the $r = 1$ regime can be ruled out so easily. Though highly improbable, perhaps some kinds of rare events can produce the kinds of time-reversed processes meeting such stringent requirements.

There is a related question as to why we live in a universe where the overwhelming proportion of processes are in the $r = 0$ regime (Bohn, 1995). This would be the case if the initial explosion of the big bang singularity was a process of type $r = 0$. All subsequent processes would then typically be of type $r = 0$ with the possible exception of exceedingly rare processes producing a type $r = 1$

event. However, the sheer preponderance of $r = 0$ processes implies an incredibly high entropy barrier that such $r = 1$ processes must overcome.

However, it does appear that the $r = 1$ regime is problematic. Under the registration/preparation arrow, observables are now represented by growing eigenvectors whereas states are represented by decaying eigenvectors. Under the excitation/de-excitation arrow, if one follows what the transformation rules suggest, de-excitations are represented by *growing* eigenvectors whereas excitations are represented by *decaying* eigenvectors. These associations are clearly not as natural as those in the $r = 0$ regime, perhaps suggesting some as yet undiscovered problems with Wigner's fourth representation. He does issue a warning that applying the extended spacetime group to unstable particles may be problematic because such particles cannot be considered as belonging to an irreducible unitary representation. But the Bohm group work utilizes nonunitary representations of *semigroups*. Rigorous work carrying out the CPT extensions of the semigroup representations has yet to be carried out (this is needed to prove that results such as vector space doubling for the group representations carry over in the semigroup case).

If there turns out to be a serious problem with this representation (e.g., a problematic unexamined assumption) such that it must be discarded, them time-reversed states would disappear from RHS quantum mechanics in the context of resonance phenomena as unphysical, leaving a purely time-asymmetric theory.

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